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Best Approximations in the Space of Bounded Linear Operators from $C(X)$ to $C(Y)$

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1. INTRODUCTION

Let X and Y be compact Hausdorff spaces, and let $C(X)$ denote the linear space of continuous bounded real-valued functions f on X, with supremum norm. The space $C(Y)$ is defined similarly. The symbol $[C(X), C(Y)]$ will denote the linear space of all bounded linear operators from $C(X)$ to $C(Y)$, with the standard operator norm given by $\| T \| = \sup\{\| T(f)\| : f \in C(X), \|f\| \leqslant 1\}$ for $T \in [C(X), C(Y)]$. If M is a subset of $[C(X), C(Y)]$ and $A \in [C(X), C(Y)]$, then a point A_0 in M is said to be a *best approximation* to A from M if $\| A - A_0 \| = \inf\{\| A - T \| : T \in M\}$. If each A in $[C(X), C(Y)]$ has a unique best approximation from M, then M is called a *Chebychev* subset of $[C(X), C(Y)]$.

This paper is concerned with the characterization of best approximations in a finite-dimensional subspace M of $[C(X), C(Y)]$, and the determination of conditions under which M is Chebychev. An element A in $[C(X), C(Y)]$ has A_0 as a best approximation in a subspace M if and only if $A - A_0$ has 0 as a best approximation in M. Therefore, to characterize best approximations in M, it suffices to provide conditions under which an element has 0 as a best approximation in M. The principal result in Section 2 provides this characterization. In Section 3, there is an investigation of finite-dimensional Chebychev subspaces of $[C(X), C(Y)]$ and a necessary condition for a finite-dimensional subspace of $[C(X), C(Y)]$ to be non-Chebychev is presented.

The problem of characterizing Chebychev subspaces for the classical Banach spaces of functions has been investigated for certain spaces. Finite-dimensional Chebychev subspaces of $C[a, b]$ have been characterized by the Haar Unicity Theorem (see, for example, [1, p. 81]). Phelps [5] has given a characterization of Chebychev subspaces of arbitrary dimension in $L_1(S, \Sigma, \mu)$

and in l_1, and has also characterized the n-dimensional Chebychev subspaces of $L_1(S, \Sigma, \mu)$ in terms of the atoms of Σ. In addition, he [4] has investigated subspaces of finite codimension in $C(X)$. In the present paper, we restrict our attention to the space of bounded linear operators from $C(X)$ to $C(Y)$.

Unless otherwise stated, notation will correspond to that of [2]. All scalars will be assumed to be real. The conjugate space $C(X)^*$ will be assumed to have the usual operator norm. For each f in $C(X)$, $\hat{f}$ will denote that functional in $C(X)^{**}$ defined by $\hat{f}(f^*) = f^*(f)$ for all f^* in $C(X)^*$, and $\hat{C}(X) = \{\hat{f} : f \in C(X)\}$. If $A_1, A_2, ..., A_n \in [C(X), C(Y)]$, then $[A_1, A_2, ..., A_n]$ will denote the linear subspace of $[C(X), C(Y)]$ spanned by these elements. We will assume, unless otherwise stated, that $[A_1, A_2, ..., A_n]$ has dimension n.

For M, a subspace of a normed linear space E with conjugate space E^*, $M^\perp = \{x^* \text{ in } E^* : x^*(x) = 0 \text{ for all } x \text{ in } M\}$. The norm closed unit sphere of E will be denoted by $S(E)$. By the weak* topology on E^*, we mean the topology on E^* obtained by taking as base all sets of the form

$$V(x^*, \hat{x}_1, ..., \hat{x}_n, \epsilon) = \{ y^* \text{ in } E^* : | \hat{x}_i(x^*) - \hat{x}_i(y^*)| < \epsilon, i = 1,...,n\}$$

for x^* in E^*, $\{x_1, ..., x_n\}$ a finite subset of E, and $\epsilon > 0$. If E and F are normed linear spaces and T is a bounded linear operator from E to F, then the adjoint T^* of T is the mapping from F^* to E^* defined by $T^*y^* = y^*T$ for y^* in F^*. By [2, p. 478] T^* is a bounded linear operator from F^* to E^*. By R^n, we will mean the space of all ordered n-tuples of real numbers with the norm of an element being the maximum of the absolute values of its components.

If Z is a normed linear space, then by $(Z \times \cdots \times Z)_\infty$ (n summands), we will mean the linear space of all ordered n-tuples of the form $z = (z_1, ..., z_n)$ for z_i in Z, $i = 1, ..., n$ with norm defined by $\| z \| = \max\{\| z_i \| : 1 \leqslant i \leqslant n\}$. The symbol $(Z \times \cdots \times Z)_1$ (n summands) is defined similarly, with the norm in this case defined by $\| z \| = \sum_{i=1}^{n} \| z_i \|$. The following lemma is then easily seen.

LEMMA 1.1. *Let Z be a normed linear space. If for $f = (f_1, ..., f_n)$ in $(Z^* \times \cdots \times Z^*)_1$ (n summands), we write $f(x_1, ..., x_n) = f_1(x_1) + \cdots + f_n(x_n)$, for all $(x_1, ..., x_n)$ in $(Z \times \cdots \times Z)_\infty$ (n summands), then*

(a) *if $E = (Z \times \cdots \times Z)_\infty$ (n summands), then E^* can be identified with $(Z^* \times \cdots \times Z^*)_1$ (n summands).*

(b) *if $E = (Z \times \cdots \times Z)_1$ (n summands), then E^* can be identified with $(Z^* \times \cdots \times Z^*)_\infty$ (n summands).*

2. Characterization of Best Approximations

In order to characterize best approximations in a finite-dimensional subspace M of $[C(X), C(Y)]$, we will need the following two lemmas. For T in $[C(X), C(Y)]$ and S a subset of Y, define T^S on $C(X)$ by

$$T^S(f) = Tf \mid S \qquad \text{for all} \quad f \text{ in } C(X),$$

where $Tf \mid S$ is the restriction of the mapping Tf to the set S.

LEMMA 2.1. *Let $A_1, \ldots, A_n$ be linearly independent operators in $[C(X), C(Y)]$ with $M = [A_1, \ldots, A_n]$. Then*

(a) *there exists a finite set $P = \{y_1, \ldots, y_p\} \subseteq Y$ such that, denoting A_i^P by $\bar{A}_i$ for $i = 1, \ldots, n$, $\{\bar{A}_1, \ldots, \bar{A}_n\}$ is a linearly independent subset of $[C(X), R^p]$.*

(b) *given B in $[C(X), C(Y)]$, there exists a non-negative constant Q such that for any finite subset S of Y with $P \subseteq S$, if $A^S = \sum_{i=1}^n \lambda_i A_i^S$ is a best approximation to B^S in $[A_1^S, \ldots, A_n^S]$, then we have $|\lambda_i| \leq Q$, $i = 1, \ldots, n$.*

Proof. For K an arbitrary finite subset of Y, define the mapping φ_K on M by $\varphi_K(A) = A^K$ where

$$A^K(f) = Af \mid K \qquad \text{for} \quad A \text{ in } M, f \text{ in } C(X).$$

Then A^K is a bounded linear operator from $C(X)$ to R^k, where k is the number of elements in K, and $\| A^K \| \leq \| A \|$. Thus φ_K is a bounded linear transformation on M and is hence continuous. We will next show that there exists a finite set $P = \{y_1, \ldots, y_p\} \subseteq Y$ such that for all A in M, $\| A \| = 1$, we have $\| \varphi_P(A) \| > \frac{1}{2}$. Let $A \in M$ with $\| A \| = 1$. Then there exists f_A in $C(X)$, $\| f_A \| \leq 1$ such that $\| Af_A \| > \frac{1}{2}$. Since Af_A is a continuous function on the compact space Y, there exists y_A in Y such that $|(Af_A(y_A))| = \| Af_A \|$. Let $K(A) = \{y_A\}$, so then $\| \varphi_{K(A)}(A) \| > \frac{1}{2}$. Let $\mathcal{O} = \{\varphi_{K(A)}(C) : C \in M$ with $\| \varphi_{K(A)}(C) \| > \frac{1}{2}\}$. Let $U(A) = \varphi_{K(A)}^{-1}(\mathcal{O})$, so $U(A)$ is open in M. Let $M' = \{A \text{ in } M : \| A \| = 1\}$. Then M' is a closed subset of $S(M)$, which is compact since M is finite-dimensional, so M' is compact. Since $\{U(A): A \in M'\}$ is an open covering of M', there exists a finite subcovering $\{U(B_1), \ldots, U(B_p)\}$ of M' for $B_1, \ldots, B_p$ in M'. Let $P = \{y_{B_1}, \ldots, y_{B_p}\}$, so P is a finite subset of Y. If $A \in M'$, then $A \in U(B_j)$ for some $j = 1, \ldots, p$. Therefore

$$\| \varphi_P(A) \| \geq \| \varphi_{K(B_j)}(A) \| > \tfrac{1}{2}.$$

Now $\varphi_P(A_i) = A_i^P = \bar{A}_i$ for $i = 1, \ldots, n$. Suppose $\bar{A}_1, \ldots, \bar{A}_n$ are linearly dependent. Then there exists A in M, $A \neq 0$, such that $\varphi_P(A) = 0$. However $A/\| A \| \in M'$, so $\| \varphi_P(A/\| A \|) \| > \frac{1}{2}$, a contradiction. Thus we must have $\bar{A}_1, \ldots, \bar{A}_n$ linearly independent, and (a) is proved.

Now let $P = \{y_1, ..., y_p\}$ be the finite subset of Y satisfying (a), and let $B \in [C(X), C(Y)]$. Denote $\varphi_P(A)$ by $\bar{A}$. Then φ_P is a continuous linear transformation from M onto $[\bar{A}_1, ..., \bar{A}_n]$ and is also one-to-one since $\bar{A}_1, ..., \bar{A}_n$ are linearly independent by (a). Thus φ_P has an inverse φ_P^{-1} which is a linear transformation. This inverse is bounded by the open mapping theorem. Now define a new norm $\| \quad \|_1$ on $[A_1, ..., A_n]$ by $\| \sum_{i=1}^n \beta_i A_i \|_1 = \max | \beta_i |$, where the maximum is taken over $1 \leqslant i \leqslant n$. Now all norms are equivalent in a finite-dimensional space, so there exists a positive contant c such that

$$\| A \|_1 \leqslant c \| A \|$$

for all A in M. Let $Q = 2c \| \varphi_P^{-1} \| \| B \|$. Let S be a finite subset of Y such that $P \subseteq S$, and let A_i^S, B^S, and A^S be as described in (b). Then $\| B^S \| \leqslant \| B \|$. It is easy to see that $\| \varphi_S^{-1} \|$ exists and $\| \varphi_S^{-1} \| \leqslant \| \varphi_P^{-1} \|$. Since A^S is a best approximation to B^S in $[A_1^S, ..., A_n^S]$, we have $\| A^S \| \leqslant 2 \| B \|$. Thus if the maximum is taken from $i = 1$ to n, we have $\max | \lambda_i | \leqslant c \| \varphi_S^{-1}(A^S) \| \leqslant Q$. This proves (b).

The preceding lemma and some of the later results utilize some techniques found in [3].

For the remainder of this section, for any set A, $\text{cl}^*(A)$ will mean the closure of A in the weak* topology.

LEMMA 2.2. *Let $E = (C(X)^* \times \cdots \times C(X)^*)_\infty$ (s summands) for s some positive integer, $A = (\hat{C}(X) \times \cdots \times \hat{C}(X))_1$ (s summands), and $M = K^\perp$ for K a finite-dimensional subspace of E. Then $A \cap M \cap S(E^*)$ is weak* dense in $M \cap S(E^*)$.*

Proof. Since $E = (C(X)^* \times \cdots \times C(X)^*)_\infty$ (s summands), E^* can be identified with $(C(X)^{**} \times \cdots \times C(X)^{**})_1$ (s summands) by Lemma 1.1 (a). Hence $A \subseteq E^*$. Since $\hat{C}(X)$ is convex, A is convex. Let E^* have the weak* topology. Suppose $K = [e_1, ..., e_k]$ for e_i in E, $i = 1, ..., k$, k some finite number. Then $M = \bigcap_{i=1}^k \{e^* \text{ in } E^* : \hat{e}_i(e^*) = 0\}$ is weak* closed. By Alaoglu's Theorem (see [2, p. 424]), $S(E^*)$ is compact in the weak* topology of E^*, and is hence weak* closed. Let $C = (C(X) \times \cdots \times C(X))_1$ (s summands). Then $C^* = E$ by Lemma 1.1 (b), so $C^{**} = E^*$. By Goldstine's Theorem (see [2, p. 424]), $\hat{S}(C)$ is weak* dense in $S(E^*)$. It is easily seen that $\hat{C} = A$. It follows that $\hat{S}(C) = A \cap S(E^*)$. Thus

$$\text{cl}^*(A \cap S(E^*)) = S(E^*).$$

The lemma can now be proven by induction on the dimension of K. Suppose K has dimension one, so that $K = [e]$ for e in E, $e \neq 0$. Clearly $\text{cl}^*(A \cap M \cap S(E^*)) \subseteq M \cap S(E^*)$. It remains to show that $M \cap S(E^*) \subseteq \text{cl}^*(A \cap M \cap S(E^*))$. Let $m \in M \cap S(E^*)$. If $m \in A$, we are finished, so assume $m \notin A$. Let U be a weak* neighborhood of m. (Without loss of generality we may take U to be a base element of the weak* topology on E^*, so U is convex.) Let $U^+ = \{e^* \text{ in } U: e^*(e) > 0\}$ and $U^- = \{e^* \text{ in } U: e^*(e) < 0\}$. Both U^+ and U^- are weak* open. We now claim that $U^+ \cap S(E^*) \neq \varnothing$. Since $e \neq 0$, we know by the Hahn–Banach Theorem (see [2, p. 62]) that there exists f^* in E^*, $\|f^*\| = 1$ such that $f^*(e) = \|e\| > 0$. Since the sequence $\{(1/n)f^* + (1 - (1/n))m\}$ in $S(E^*)$ converges to m in the weak* topology on E^*, there exists a positive integer N such that $g^* = (1/N)f^* + (1 - (1/N))\, m \in U$. Then $g^* \in U^+ \cap S(E^*)$. Similarly, $U^- \cap S(E^*) \neq \varnothing$. Then since $S(E^*) = \mathrm{cl}^*(A \cap S(E^*))$, there must exist f_1^* in $A \cap S(E^*) \cap U^+$. Similarly, there exists f_2^* in $A \cap S(E^*) \cap U^-$. Then there exists λ in $(0, 1)$ such that $\lambda f_1^*(e) + (1 - \lambda) f_2^*(e) = 0$. Let $e^* = \lambda f_1^* + (1 - \lambda) f_2^*$. Then $e^* \in A$, $S(E^*)$, and U, since each of these sets is convex. Therefore we have exhibited e^* in $A \cap M \cap S(E^*)$, e^* in U, and $e^* \neq m$. Thus, $m \in \mathrm{cl}^*(A \cap M \cap S(E^*))$, completing the proof for the case when K has dimension one.

Now suppose the lemma holds for a k-dimensional subspace of E. Let $K = [e_1, ..., e_{k+1}]$ for $e_1, ..., e_{k+1}$ in E, so K has dimension $k + 1$. Then for $M = K^\perp$, clearly $\mathrm{cl}^*(A \cap M \cap S(E^*)) \subseteq M \cap S(E^*)$. Now let $m \in M \cap S(E^*)$, $m \notin A$, and let U be a convex weak* neighborhood of m. Let $K' = [e_1, ..., e_k]$ and $M' = K'^\perp$. Then $m \in M' \cap S(E^*) = \mathrm{cl}^*(A \cap M' \cap S(E^*))$ by the hypothesis of induction. Letting $U^+ = \{e^* \text{ in } U : e^*(e_{k+1}) > 0\}$ and $U^- = \{e^* \text{ in } U : e^*(e_{k+1}) < 0\}$, we then utilize the Hahn–Banach Theorem to obtain f^* in E^*, $\|f^*\| = 1$ with $f^*(K') = 0$ and $f^*(e_{k+1}) > 0$. Proceeding in a manner analogous to that of the one-dimensional case, we see that $M' \cap S(E^*) \cap U^+$ and $M' \cap S(E^*) \cap U^-$ are nonempty sets. Then since $M' \cap S(E^*) = \mathrm{cl}^*(A \cap M' \cap S(E^*))$, the procedure of the one-dimensional case will lead us to $\mathrm{cl}^*(A \cap M \cap S(E^*)) = M \cap S(E^*)$ for K of dimension $k + 1$. This completes the induction and the proof.

Our main theorem here is the following characterization of best approximations, in which we give necessary and sufficient conditions for an element to have 0 as a best approximation in a finite-dimensional subspace of $[(C(X), C(Y)]$. Without loss of generality, we may assume that each of the operators generating the subspace has norm 1.

THEOREM 2.3. *Let $A_k \in [C(X), C(Y)]$ with $\|A_k\| = 1$, $k = 1, ..., n$, and let $B \in [C(X), C(Y)]$. Then B has 0 as a best approximation in $[A_1, ..., A_n]$ if and only if for all $\epsilon > 0$, there exist m elements of Y, $y_1, ..., y_m$, m functions $f^1, ..., f^m$ in $C(X)$ with $\|f^i\| \leqslant 1$, $i = 1, ..., m$, and m scalars $r_1, ..., r_m$ with $r_i > 0$, $i = 1, ..., m$ and $\sum_{i=1}^m r_i = 1$ such that*

(i) $\sum_{i=1}^m r_i(A_k f^i)(y_i) = 0$ *for* $k = 1, ..., n$

(ii) $\left| \sum_{i=1}^m r_i(Bf^i)(y_i) - \|B\| \right| < \epsilon$.

Proof. Necessity. Choose P and Q as in Lemma 2.1. Suppose B has 0 as a best approximation in $[A_1, ..., A_n]$. Let $\lambda_1, ..., \lambda_n \in [-Q, Q]$. To simplify notation, let $T_\lambda = B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)$. Let $\hat{y}$ be the point evaluation function in $C(Y)^*$ defined by $\hat{y}(f) = f(y)$ for all f in $C(Y)$. Let S denote the closed unit sphere of $C(Y)^*$. Note that $\| T_\lambda^* \| = \sup \| T_\lambda^*(\mu)\|$ where the supremum is taken over all μ in S. Then the extreme points of S are given by ext $S = \{\pm\hat{y} : y \text{ in } Y\}$ by [2, p. 441]. We know S is compact in the weak* topology of $C(Y)^*$ by Alaoglu's Theorem (see [2, p. 424]) and is also convex. Thus, by the Krein–Milman Theorem (see [2, p. 440]), $S = \text{cl}^*(\text{co}(\text{ext } S))$, where for any set A, $\text{cl}^*(\text{co}(A))$ denotes the closed convex hull of A in the weak* topology.

Let $\epsilon > 0$. We will show that there exists $y_\lambda = y(\lambda_1, ..., \lambda_n)$ in Y such that $| \| T_\lambda^*(\hat{y}_\lambda)\| - \| T_\lambda^* \| | < (\epsilon/6)$. Suppose not. Then $\sup\{\| T_\lambda^*(\hat{y})\| : y \text{ in } Y\} = L < \| T_\lambda^* \|$. Now T_λ^* is a weak* continuous linear transformation from $C(Y)^*$ into $C(X)^*$ by [2, p. 478]. Since T_λ^* maps $\{\hat{y} : y \text{ in } Y\}$ into the weak* compact convex set $S_L = \{\nu \in C(X)^* : \| \nu \| \leqslant L\}$, it maps ext S and hence all of S into S_L, which implies $\| T_\lambda^* \| = L$. By this contradiction, it follows that

$$| \| T_\lambda^* \hat{y}_\lambda \| - \| T_\lambda \| | < (\epsilon/6). \tag{2.1}$$

Now let $\mu_1, ..., \mu_n \in [-Q, Q]$. It is easily seen that the function

$$\varphi(\mu_1, ..., \mu_n) = \| T_\mu^* \hat{y}_\lambda \|$$

is continuous at $(\lambda_1, ..., \lambda_n)$. Hence for $\epsilon/6$, for each $i = 1, ..., n$ there exists an open interval $I_{\lambda_i} = \{\mu : | \mu - \lambda_i | < (\epsilon/6n)\}$ such that for $\mu_1, ..., \mu_n \in [-Q, Q]$, if $\mu_i \in I_{\lambda_i}$ for each $i = 1, ..., n$ then

$$| \| T_\mu^* \hat{y}_\lambda \| - \| T_\lambda^* \hat{y}_\lambda \| | < (\epsilon/6). \tag{2.2}$$

Using (2.1) and (2.2), we obtain $| \| T_\mu^* \hat{y}_\lambda \| - \| T_\mu \| | < (\epsilon/2)$. Thus we have shown for $\mu_1, ..., \mu_n \in [-Q, Q]$ and $\mu_i \in I_{\lambda_i}$, $i = 1, ..., n$ that (taking the supremum over all f in $C(X)$), $\|f\| \leqslant 1$, we have

$$| \sup | T_\mu f(y_\lambda)| - \| T_\mu \| | < (\epsilon/2). \tag{2.3}$$

For a scalar λ in $[-Q, Q]$, let $I_\lambda = \{\mu : | \mu - \lambda | < (\epsilon/6n)\}$. Then $\{I_\lambda : \lambda \text{ in } [-Q, Q]\}$ is an open covering of the compact set $[-Q, Q]$. Therefore, there exists a finite number of scalars $\lambda_1, ..., \lambda_s$ in $[-Q, Q]$ such that $\{I_{\lambda_j} : j = 1, ..., s\}$ is also a covering of $[-Q, Q]$. Recall that for $\lambda_1, ..., \lambda_n$ in $[-Q, Q]$, y_λ is selected so that (2.1) holds. Consider $y_{\lambda(p)} = y(\lambda_{p(1)}, ..., \lambda_{p(n)})$ in Y where

$\lambda_{p(i)}$ may be chosen from λ_1 to λ_s for $i = 1,\ldots, n$. Let P' be the set of these s^n elements of Y, and let F be the union of the sets P' and P. Let m be the number of distinct points in F, and label these points $y_1 ,\ldots, y_m$, so $F = \{y_1 ,\ldots, y_m\}$ is a subset of Y.

Let $\lambda_1 ,\ldots, \lambda_n$ be arbitrary scalars in $[-Q, Q]$. Then for each $i = 1,\ldots, n$, $\lambda_i \in I_{\lambda_{j(i)}}$ for some $j(i) = 1,\ldots, s$, so that $| \lambda_i - \lambda_{j(i)} | < (\epsilon/6n)$ for $i = 1,\ldots, n$. Now $y_{\lambda_j} = y(\lambda_{j(1)} ,\ldots, \lambda_{j(n)}) = y_j$ for some $j = 1,\ldots, m$. Then by (2.3), taking the supremum over all f in $C(X)$, $\|f\| \leqslant 1$, we have

$$| \sup | T_\lambda f(y_{\lambda_j})| - \| T_\lambda \|| < (\epsilon/2).$$

Thus

$$| \max \sup | T_\lambda f(y)| - \| T_\lambda \| | < (\epsilon/2), \tag{2.4}$$

where the maximum is taken over all y in F and the supremum runs over all f in $C(X)$, $\|f\| \leqslant 1$.

Let $T \in [C(X), C(Y)]$. For f in $C(X)$, define $\bar{T}$ by $\bar{T}f = Tf | F$. Then $\bar{T}$ is a bounded linear operator from $C(X)$ to R^m with $\| \bar{T} \| \leqslant \| T \|$. Now for $| \lambda_i | \leqslant Q$, $i = 1,\ldots, n$, it follows from (2.4) that

$$| \| \bar{B} - (\lambda_1\bar{A}_1 + \cdots + \lambda_n\bar{A}_n)\| - \| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\| | < (\epsilon/2). \tag{2.5}$$

Let $E = (C(X)^* \times \cdots \times C(X)^*)_\infty$ (m summands), so $\bar{T} \in E$ for T in $[C(X), C(Y)]$. Consider the quotient space $E/[\bar{A}_1 ,\ldots, \bar{A}_n]$ with the quotient mapping $\pi : E \to E/[\bar{A}_1 ,\ldots, \bar{A}_n]$. By Lemma 2.1 (b),

$$\| \pi\bar{B} \| = \inf \| \bar{B} - (\lambda_1\bar{A}_1 + \cdots + \lambda_n\bar{A}_n)\|$$

where the infimum is taken over all λ_i in $[-Q, Q]$, $i = 1,\ldots, n$. Now

$$\| \pi\bar{B} \| \leqslant \| B \|.$$

Then since B has 0 as a best approximation in $[A_1 ,\ldots, A_n]$, it follows from (2.5) that

$$| \| \pi\bar{B} \| - \| B \| | < (2\epsilon/3). \tag{2.6}$$

Suppose $B \neq 0$. Then by the Hahn–Banach Theorem (see [2, p. 62]) there exists H in E^*, $\| H \| = 1$, such that $H([\bar{A}_1 ,\ldots, \bar{A}_n]) = 0$ and $H(\bar{B}) = \| \pi\bar{B} \|$. By (2.6),

$$| H(\bar{B}) - \| B \| | < (2\epsilon/3). \tag{2.7}$$

Let $A = (\hat{C}(X) \times \cdots \times \hat{C}(X))_1$ (m summands), $K = [\bar{A}_1 ,\ldots, \bar{A}_n]$, and $M = K^\perp$. Then by Lemma 2.2, we have $A \cap M \cap S(E^*)$ is weak* dense in

$M \cap S(E^*)$. Consider $V = V(H, \hat{\bar{B}}, (\epsilon/3))$, a weak* neighborhood of H. Then there exists G in V such that $G \in A \cap M \cap S(E^*)$. Thus

$$G([\bar{A}_1, ..., \bar{A}_n]) = 0. \tag{2.8}$$

Since $| H(\bar{B}) - G(\bar{B})| < (\epsilon/3)$, it follows from (2.7) that

$$| G(\bar{B}) - \| B \| | < \epsilon. \tag{2.9}$$

Now G can be represented by $(\hat{g}^1, ..., \hat{g}^m)$ where $g^i \in C(X)$ for $i = 1, ..., m$ and $\| G \| = \sum_{i=1}^m \| \hat{g}^i \| \leqslant 1$. Without loss of generality, assume $\| g^i \| > 0$ for $i = 1, ..., m$. Define r_i in the following way:

$$r_i = \begin{cases} \| g^i \| & \text{if} \quad i = 1, ..., m-1 \\ 1 - \sum_{i=1}^{m-1} \| g^i \| & \text{if} \quad i = m. \end{cases}$$

Then $r_i > 0$ for $i = 1, ..., m$ and $\sum_{i=1}^m r_i = 1$. Now define f^i by

$$f^i = \begin{cases} g^i/\| g^i \| & \text{if} \quad i = 1, ..., m-1 \\ g^i \Big/ \left(1 - \sum_{i=1}^{m-1} \| g^i \|\right) & \text{if} \quad i = m. \end{cases}$$

Then $f^i \in C(X)$ with $\| f^i \| \leqslant 1$, $i = 1, ..., m$. For T in $[C(X), C(Y)]$, we know $\bar{T} \in E$, so $\bar{T}$ can be represented by $(T^*\hat{y}_1, ..., T^*\hat{y}_m)$. A simple computation shows that $G(\bar{T}) = \sum_{i=1}^m r_i(Tf^i)(y_i)$. Conditions (i) and (ii) hold by (2.8) and (2.9).

If $B = 0$, the result is established by taking $m = 1$, y an arbitrary element of Y, $f = 0$ in $C(X)$, and $r = 1$.

Sufficiency. Let $\epsilon > 0$. Then there exist m elements of Y, $y_1, ..., y_m$, m functions $f^1, ..., f^m$ in $C(X)$ with $\| f^i \| \leqslant 1$, $i = 1, ..., m$, and m scalars $r_1, ..., r_m$ with $r_i > 0$, $i = 1, ..., m$ and $\sum_1^m r_i = 1$ such that (i) and (ii) hold. For T in $[C(X), C(Y)]$, define G by

$$G(T) = \sum_1^m r_i(Tf^i)(y_i).$$

For $k = 1, ..., n$, $G(A_k) = 0$ by (i), and $| G(B) - \| B \| | < \epsilon$ by (ii). Now G is a bounded linear functional on $[C(X), C(Y)]$ with $\| G \| \leqslant 1$. Let $\lambda_1, ..., \lambda_n$ be arbitrary scalars. Then $| G(B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)) - \| B \| | < \epsilon$. Hence $\| B \| - \epsilon < \| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\|$. But this can be shown for all $\epsilon > 0$. Therefore $\| B \| \leqslant \| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\|$, so B has 0 as a best approximation in $[A_1, ..., A_n]$. This completes the proof.

3. Finite-Dimensional Chebychev Subspaces

In this section we consider an arbitrary finite-dimensional subspace M of $[C(X), C(Y)]$, and in the first theorem we present a necessary condition for M to be non-Chebychev.

THEOREM 3.1. *Let $M = [A_1, ..., A_n]$ be a non-Chebychev subspace of $[C(X), C(Y)]$ with $\| A_k \| = 1, k = 1, ..., n$. Then there exists A in M, $\| A \| = 1$ such that given $\epsilon > 0$, there exist m elements $y_1, ..., y_m$ in Y and m functions $f^1, ..., f^m$ in $C(X)$ with $\sum_{i=1}^m \| f^i \| \leqslant 1$ such that if α in $[C(X), C(Y)]^*$ is defined by $\alpha(T) = \sum_{i=1}^m Tf^i(y_i)$, then*

(i) $\alpha \in M^\perp$

(ii) *if $\beta \in [C(X), C(Y)]^*$ and $\| \alpha \pm \beta \| \leqslant 1$, then $| \beta(A)| < \epsilon$*

(iii) $\sum_{i=1}^m | Af^i(y_i)| < \epsilon$.

Proof. Since M is non-Chebychev, it follows that there exists B in $[C(X), C(Y)]$ such that B has 0 and $\pm A \neq 0$ as best approximations in M with $\| A \| = 1$. Then $\| B \| = \| B - A \| = \| B + A \|$. This will be the required A. Let $\epsilon > 0$. Since B has 0 as a best approximation in M, by Theorem 2.3 for $(\epsilon/2) > 0$, there exist m elements of Y, $y_1, ..., y_m$, m functions $h^1, ..., h^m$ in $C(X)$ with $\| h^i \| \leqslant 1$, $i = 1, ..., m$, and m scalars $r_1, ..., r_m$ with $r_i > 0$, $i = 1, ..., m$ and $\sum_1^m r_i = 1$ such that

(i′) $\sum_1^m r_i(A_k h^i)(y_i) = 0 \qquad \text{for} \qquad k = 1, ..., n$

(ii′) $| \sum_1^m r_i(Bh^i)(y_i) - \| B \| | < (\epsilon/2)$.

For $i = 1, ..., m$, let $f^i = r_i h^i$. Then $f^i \in C(X)$, $i = 1, ..., m$ and $\sum_1^m \| f^i \| \leqslant 1$. Define α on $[C(X), C(Y)]$ by $\alpha(T) = \sum_{i=1}^m Tf^i(y_i)$ for T in $[C(X), C(Y)]$. Then $\alpha \in [C(X), C(Y)]^*$. By (i′), $\alpha \in M^\perp$ and by (ii′) we obtain

$$| \alpha(B) - \| B \| | < (\epsilon/2). \tag{3.1}$$

To prove (ii), let $\beta \in [C(X), C(Y)]^*$ with $\| \alpha \pm \beta \| \leqslant 1$. Then $(\alpha \pm \beta)(B) \leqslant \| B \|$. Since $\alpha \in M^\perp$, $\alpha(B) \pm \beta(B - A) \leqslant \| B - A \|$. Using (3.1), we obtain $| \beta(B)| < (\epsilon/2)$ and $| \beta(B - A)| < (\epsilon/2)$. Hence $| \beta(A)| < \epsilon$.

We must now show (iii). Let $P = \{i : Af^i(y_i) \geqslant 0\}$, $P' = \{i : Af^i(y_i) > 0\}$ and $N = \{i : Af^i(y_i) < 0\}$. Since $\alpha \in M^\perp$, $\sum_1^m Af^i(y_i) = 0$. Thus, if either one of P' or N is empty, so is the other, and (iii) clearly holds. Therefore, assume both P' and N are nonempty. Since $\sum_{i \in P} Af^i(y_i) + \sum_{i \in N} Af^i(y_i) = 0$, we must have $\sum_{i \in P} | Af^i(y_i)| = \sum_{i \in N} | Af^i(y_i)|$. Now suppose (iii) is false. Then

$$\sum_{i \in P} | Af^i(y_i)| \geqslant (\epsilon/2) \tag{3.2}$$

and $\sum_{i \in N} | Af^i(y_i)| \geqslant (\epsilon/2)$. Let $\lambda_i = \|f^i\|$ for each $i = 1,..., m$. Let $\lambda_P = \sum_{i \in P} \lambda_i > 0$ and $\lambda_N = \sum_{i \in N} \lambda_i > 0$. Then $\lambda_P + \lambda_N \leqslant 1$. Let $S_1 = \sum_{i \in P} Bf^i(y_i)$ and $S_2 = \sum_{i \in N} Bf^i(y_i)$. Then using (3.1), we have

$$S_1 + S_2 > \| B \| - (\epsilon/2).$$

Thus either (a) $S_1 > \lambda_P(\| B \| - (\epsilon/2))$ or (b) $S_2 > \lambda_N(\| B \| - (\epsilon/2))$ must hold. Suppose (a) holds. Then since $\lambda_P < 1$, by (3.2) we see that

$$\sum_{i \in P} (B + A) f^i(y_i) > \lambda_P \| B + A \| .$$

But for each i in P, $(B + A) f^i(y_i) \leqslant \lambda_i \| B + A \|$. By summing both sides over all i in P, we are led to a contradiction. If (b) is true, a similar argument using N and $B - A$ provides a contradiction. Thus, (iii) is proved.

We conclude this section with the following result.

THEOREM 3.2. *Let $M = [A_1 ,..., A_n]$ be an n-dimensional Chebychev subspace of $[C(X), C(Y)]$, and let K be a subset of Y which is both open and closed in Y. For T in $[C(X), C(Y)]$, define $\bar{T} : C(X) \to C(K)$ by $\bar{T}f = Tf \mid K$. Let $\bar{M} = \{\bar{T}: T \text{ in } M\}$. Then $\bar{M}$ is a Chebychev subspace of $[C(X), C(K)]$ and is n-dimensional if $[C(X), C(K)]$ has dimension $\geqslant n$. (This happens, in particular, if either X or K has n or more points.)*

Proof. First we will prove the following claim, which will be denoted by (3.3). Let $T \in [C(X), C(K)]$, $\| T \| = 1$, such that T has 0 as a best approximation in $\bar{M}$. Then if $A \in M$, $A \neq 0$, $\| A \| \leqslant 1$, we must have $\| T \| < \| T - \bar{A} \|$. Suppose the claim is false. Then there exists A in M, $A \neq 0$, $\| A \| \leqslant 1$ such that $\| T \| = \| T - \bar{A} \|$. For each f in $C(X)$, extend Tf to all of Y by defining $Tf(y) = 0$ for $y \notin K$. With this extension, $T \in [C(X), C(Y)]$. Define B on $C(X)$ in the following way: for f in $C(X)$, let

$$Bf(y) = \begin{cases} Tf(y) & \text{if } \; y \in K \\ Af(y) & \text{if } \; y \notin K. \end{cases}$$

Then $B \in [C(X), C(Y)]$ with $\| B \| = 1$. Now let $C \in M$. Since T has 0 as a best approximation in $\bar{M}$, it follows that $\| T - \bar{C} \| \geqslant 1$. Then for $\epsilon > 0$ there exists f_0 in $C(X)$, $\| f_0 \| \leqslant 1$, and y_0 in K such that

$$|(T - \bar{C}) f_0(y_0)| > 1 - \epsilon.$$

Thus $\| B - C \| > 1 - \epsilon$. Hence, $\| B - C \| \geqslant \| B \|$, so B has 0 as a best approximation in M. Since for any f in $C(X)$, $(B - A) f(y) = 0$ if $y \notin K$, it can be easily shown that $\| B - A \| = \| B \|$. But this is impossible, since M is Chebychev. Therefore the claim (3.3) is proved.

Now suppose $\overline{M}$ is not Chebychev. Then, it follows that there exists T in $[C(X), C(K)]$, $\| T \| = 1$, such that T has 0 and $\bar{A} \neq 0$ as best approximations in $\overline{M}$, where $A \in M$. We may assume $\| A \| \leqslant 1$ by the convexity of the set of best approximations to T in $\overline{M}$. Then $\| T \| = \| T - \bar{A} \|$, which contradicts (3.3). Therefore $\overline{M}$ is Chebychev.

Suppose $[C(X), C(K)]$ has dimension $\geqslant n$. We can now show that $\overline{M}$ is n-dimensional. Suppose not. Then $\overline{M} \neq [C(X), C(K)]$. Let $A \in M$ with $\bar{A} = 0$. Without loss of generality assume $\| A \| = 1$. Select T as in claim (3.3). It is then easy to see by this claim that $A = 0$. Now $\bar{A}_1 ,..., \bar{A}_n$ must be linearly dependent, so there exist scalars $\lambda_1 ,..., \lambda_n$ not all 0 such that

$$\lambda_1 \bar{A}_1 + \cdots + \lambda_n \bar{A}_n = 0.$$

Let $A = \lambda_1 A_1 + \cdots + \lambda_n A_n$. Then $A \in M$ with $\bar{A} = 0$, so $A = 0$. But this is impossible, since M has dimension n. Therefore $\overline{M}$ is an n-dimensional subspace of $[C(X), C(K)]$.

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